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On the method of Horace for free resolutions in $\mathbb{P}^n$

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Abstract: **In this paper, we show how the method of Horace can be used to prove maximal rank hypothesis in $\mathbb{P}^n$. This can be useful in finding generators of the homogeneous ideals in $\mathbb{P}^n$; results which can be applied in the proof of the minimal resolution conjecture.**

Keywords: **method of Horace, ideal of general points, minimal free resolutions.**

1. INTRODUCTION

The minimal resolution conjecture asserts that the minimal free resolution of a set M of m points in general position has no ghost terms. That is, the conjecture gives the prescribed form of the minimal free resolution of these m points.

The minimal resolution conjecture predicts the correct form of minimal free resolution for general points in $\mathbb{P}^n$ for $n \leq 4$. This has been proved in [5], [6], [2], [3], [9], [10] and [11]. In $\mathbb{P}^n$, where $n \geq 6$, counter examples to the conjecture have been found (see [4]), except for $n = 5$ and $n = 9$. These counter examples appear for as few as 11 points in $\mathbb{P}^6$. In $\mathbb{P}^9$, these counter examples have not appeared for 50 or fewer points. In general, the conjecture holds true when the number of points, m, is sufficiently large compared to the dimension of the projective space, that is, $m >> n$ (see [7]).

Our aim is to show how the method of Horace can be used to find generators of homogeneous ideals associated with the szygies in the free resolution of points in general position in $\mathbb{P}^n$.

The paper is organized as follows. We will first give some results that we will use in our main work and also build notation and develop the language used. We will then describe the method of Horace and put it in the context of minimal free resolution for $\mathbb{P}^n$before showing how this method can be used to tackle the conjecture in $\mathbb{P}^n$.

2. PRELIMINARIES

The minimal resolution conjecture was formulated by Lorenzini [1], and it predicts the form the minimal free resolution for the ideal of generic points in the projective space. Suppose

$M = \{P_1, P_2, \cdots P_m\}$, where $m \geq n + 1$, is a set of points in general position. Let X be the sub-scheme supported at these points. Then the homogeneous ideal $I_X \subseteq R = k[x_0, x_1, \cdots x_n]$ where k is an algebraically closed field and R the homogeneous coordinate ring of $\mathbb{P}^n$, has the following form;

$$0 \longrightarrow F_{n-1} \ldots \longrightarrow F_p \ldots \longrightarrow F_0 \longrightarrow I_X \longrightarrow 0$$

If in addition the number m of points are in the d$^{\text{th}}$ binomial interval and also satisfy $m \leq h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$, then the points impose independent conditions and each module F_p is a direct sum of copies of degree $d + p$ generators and degree $d + p + 1$ generators. Also the minimal resolution conjecture gives a relation among the degree $d + p$ generators in F_{p+1} and F_p. More precisely, we have that each of the module F_p is of the form $F_p = R(-d - p)^{a_{p-1}} \oplus R(-d - p - 1)^{b_p}$, with the non-negative integers a_p and b_p called the graded Betti numbers, satisfying;

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$$a_p = max\left\{0, h^0\left(\mathbb{P}^n,\ \Omega_{\mathbb{P}^n}^{p+1}(d+p+1)\right) - rk\left(\Omega_{\mathbb{P}^n}^{p+1}(d+p+1)\right)m\right\}$$

And $b_p = max\left\{0, rk\left(\Omega_{\mathbb{P}^n}^{p+1}(d+p+1)\right)m - h^0\left(\mathbb{P}^n,\ \Omega_{\mathbb{P}^n}^{p+1}(d+p+1)\right)\right\}$

It has been shown in [7] that the problem of existence of the minimal free resolution of the form above can be reduced to proving that the evaluation map below is of maximal rank for all

$0 \leq p \leq n-2$.

$$H^0\left(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^{p+1}(d+p+1)\right) \longrightarrow \oplus_{i=1}^{m} \Omega_{\mathbb{P}^n}^{p+1}(d+p+1)|_{P_i}$$

This is the same as saying that the betti numbers a_p and b_p satisfy $a_p b_p = 0$ for $p = 0, 1, 2, \cdots n-2$.

Method of Horace

The method of Horace is presented in [7]. It is an inductive method that makes use of elementary transformation of vector bundles.

Theorem 2.1 (Method of Horace)

Suppose we are given a surjective morphism of vector spaces,

$$\lambda: H^0(X', F') \longrightarrow L$$

and suppose that there exist a point $Z' \in X'$such that

$$\lambda: H^0(X', F') \hookrightarrow L \oplus F'$$

and suppose that $H^1(X, E) = 0$. Then there exist a quotient $E(Z') \longrightarrow D(\lambda)$ with a kernel contained in $F'(Z')$ of dimension $dim(D(\lambda)) = rk(F) - dim(ker\ \lambda)$ having the following property: Let $\mu: H^0(X, F) \longrightarrow N$ be a morphism of vector spaces, then there exist $Z \in X'$ such that if $H^0(X, E) \longrightarrow N \oplus D(\lambda)$ is of maximal rank then $H^0(X, F) \longrightarrow N \oplus L \oplus F(Z)$ is also of maximal rank.

The idea of the theorem is illustrated in the diagram below.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^0(X,E) & \longrightarrow & \mathrm{H}^0(X,F) & \longrightarrow & \mathrm{H}^0(X',F') & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \longrightarrow & N \oplus \mathrm{D}(\lambda) & \longrightarrow & N \oplus L \oplus F_{|Z} & \longrightarrow & L \oplus \mathrm{D}'(\lambda)_{|Z} & \longrightarrow & 0 \end{array}$$

The key point is that if the map γ is bijective, then β will be bijective provided that α is bijective.

Remark 2.2

The method of Horace involves application of theorem 2.1 and diagram chasing. The diagram chasing is done on exact sequences obtained from elementary transformation of vector bundles.

3. METHOD OF HORACE FOR $\mathbb{P}^n$

In this section, we show how the method of Horace can be used in verifying the maximal rank hypothesis. We begin by giving an elementary transformation of vector bundles on $\mathbb{P}^n$ and identify the sequences will be used in the diagram chasing. With these sequences, we will put theorem 2.1 in our context. We will then formulate the inductive hypotheses and give the lemmas which when proved to be true proves the inductive hypotheses. The folloing elementary transformation will be used.

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$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Omega^{p+1}_{\mathbb{P}^n}(d+p) & = & \Omega^{p+1}_{\mathbb{P}^n}(d+p) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}(d-1)^{\oplus\binom{n}{p+1}} & \longrightarrow & \Omega^{p+1}_{\mathbb{P}^n}(d+p+1) & \longrightarrow & \Omega^{p+1}_{\mathbb{P}^{n-1}}(d+p+1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Omega^{p}_{\mathbb{P}^{n-1}}(d+p) & \longrightarrow & \Omega^{p+1}_{(\mathbb{P}^n|\mathbb{P}^{n-1})}(d+p+1) & \longrightarrow & \Omega^{p+1}_{\mathbb{P}^{n-1}}(d+p+1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{3.1}$$

Remark 3.1

To put the method of Horace in the context, we set $X = \mathbb{P}^n$, $X' = \mathbb{P}^{n-1}$, $F = \Omega^{p+1}_{\mathbb{P}^n}$, $E = \mathcal{O}_{\mathbb{P}^n}^{\oplus\binom{n}{p+1}}$ and $F' = \Omega^{p+1}_{\mathbb{P}^{n-1}}$. We use the middle sequence and the leftmost sequence to. As an illustration using the middle sequence, theorem 2.1 asserts that if the map γ in the diagram below is bijective, then β will be bijective provided that α is bijective.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-1)^{\oplus\binom{n}{p+1}}\right) & \longrightarrow & \mathrm{H}^0\left(\mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^n}(d+p+1)\right) & \longrightarrow & \mathrm{H}^0\left(\mathbb{P}^{n-1}, \Omega^{p+1}_{\mathbb{P}^{n-1}}(d+p+1)\right) & \longrightarrow & 0 \\
 & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
0 & \longrightarrow & N \oplus \mathrm{D}(\lambda) & \longrightarrow & N \oplus L \oplus F_{(\ \ |Z)} & \longrightarrow & L \oplus \mathrm{D}'(\lambda)_{(\ \ |Z)} & \longrightarrow & 0
\end{array}$$

The method of Horace involves defining the evaluation maps, proving all possible cases of N, L and the quotients. For this particular case, proving these cases is the same as proving the following inductive hypotheses.

Hypothesis 3.2 $\mathbf{H}\left(\boldsymbol{\Omega}^{\mathbf{p+1}}_{\mathbb{P}^n}(\boldsymbol{d+p+1}), \boldsymbol{\Omega}^{\mathbf{p+1}}_{\mathbb{P}^{n-1}}(\boldsymbol{d+p+1});\ \boldsymbol{a, b, c}\right)$

The statement $\mathrm{H}\left(\Omega^{p+1}_{\mathbb{P}^n}(d+p+1), \Omega^{p+1}_{\mathbb{P}^{n-1}}(d+p+1);\ a, b, c\right)$ asserts that for all non-negative integers a, b, c and θsatisfying, where θ stands for the dimension of the quotient and satisfy $\theta \leq \binom{n}{p+1}$, we have that the map

$$\mathrm{H}^0\left(\mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^n}(d+p+1)\right) \longrightarrow \oplus_{i=1}^{a} \Omega^{p+1}_{\mathbb{P}^n}(d+p+1)|_{A_1} \oplus_{j=1}^{b} \Omega^{p+1}_{\mathbb{P}^n}(d+p+1)|_{B_j} \oplus \Gamma|_C$$

is bijective. Here $\binom{n}{p+1} a + \binom{n-1}{p+1} b + \theta c = h^0\left(\mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^n}(d+p+1)\right)$.

Hypothesis 3.3 $\mathbf{H}\left(\boldsymbol{\mathcal{O}}_{\mathbb{P}^n}(\mathbf{d-1})^{\oplus\binom{\mathbf{n}}{\mathbf{p+1}}}, \boldsymbol{\Omega}^{\mathbf{p}}_{\mathbb{P}^{n-1}}(\boldsymbol{d+p});\ \boldsymbol{e, f, g}\right)$

The statement $\mathrm{H}\left(\mathcal{O}_{\mathbb{P}^n}(d-1)^{\oplus\binom{n}{p+1}}, \Omega^{p}_{\mathbb{P}^{n-1}}(d+p);\ e, f, g\right)$ asserts that for all non-negative integers e, f, g and εsatisfying, where ε stands for the dimension of the quotient, we have that the map

$$\mathrm{H}^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-1)^{\oplus\binom{n}{p+1}}\right) \longrightarrow \oplus_{i=1}^{e} \mathcal{O}_{\mathbb{P}^n}(d-1)^{\oplus\binom{n}{p+1}}|_{A_1} \oplus_{j=1}^{f} \mathcal{O}_{\mathbb{P}^{n-1}}(d-1)^{\oplus\binom{n}{p+1}}|_{F_j} \oplus \Gamma|_G$$

is bijective. Here $\binom{n}{p+1} e + \binom{n-1}{p} f + \varepsilon g = h^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-1)^{\oplus\binom{n}{p+1}}\right)$.

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Remark 3.4

a. In order to use the method of Horace in proving maximal rank hypothesis for a particular n, one proves the above hypotheses for $0 \leq p \leq n-2$. The induction is done on d while fixing p.

b. The proof of the hypotheses above follows from the verification of the following lemmas. The lemmas form the inductive part of the proof. The base step involves proving by hand the maximal rank hypothesis for the smallest possible cases.

Lemma 3.5

Suppose d, a, b and c are non-negative integers satisfying the conditions of hypothesis 3.2. Write

$\mathrm{h}^0\left(\mathbb{P}^{n-1}, \Omega^{\mathrm{p}+1}_{\mathbb{P}^{n-1}}(d+p+1)\right) - \binom{n-1}{p+1} b - \theta c = \binom{n-1}{p} f + \varepsilon g$. Set $e = a - f - g$. If $a \geq 0$ and $\binom{n-1}{p} f + \varepsilon g \leq \mathrm{h}^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\mathrm{d}-1)^{\oplus\binom{\mathrm{n}}{\mathrm{p}+1}}\right)$, then $\mathrm{H}\left(\mathcal{O}_{\mathbb{P}^n}(\mathrm{d}-1)^{\oplus\binom{\mathrm{n}}{\mathrm{p}+1}}, \Omega^{\mathrm{p}}_{\mathbb{P}^{n-1}}(d+p);\ e, f, g\right)$ implies $\mathrm{H}\left(\Omega^{\mathrm{p}+1}_{\mathbb{P}^n}(d+p+1), \Omega^{\mathrm{p}+1}_{\mathbb{P}^{n-1}}(d+p+1);\ a, b, c\right)$.

Proof. The proof of this lemma follows from application of theorem 2.1 on the exact sequence

$$0 \longrightarrow \mathrm{H}^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\mathrm{d}-1)^{\oplus\binom{\mathrm{n}}{\mathrm{p}+1}}\right) \longrightarrow \mathrm{H}^0\left(\mathbb{P}^n, \Omega^{\mathrm{p}+1}_{\mathbb{P}^n}(d+p+1)\right) \longrightarrow \mathrm{H}^0\left(\mathbb{P}^{n-1}, \Omega^{\mathrm{p}+1}_{\mathbb{P}^{n-1}}(d+p+1)\right) \longrightarrow 0$$

Lemma 3.7

Suppose d, e, f and g are non-negative integers satisfying the conditions of hypothesis 3.3. Write $\mathrm{h}^0\left(\mathbb{P}^{n-1}, \Omega^{\mathrm{p}}_{\mathbb{P}^{n-1}}(d+p)\right) - \binom{n-1}{p} f - \varepsilon g = \binom{n-1}{p+1}\bar{b} - \bar{\theta}\bar{c}$, where $\bar{b} \geq 0$. Set $\bar{a} = e - \bar{b} - \bar{c}$. If $\bar{a} \geq 0$ and $\binom{n-1}{p+1}\bar{b} - \bar{\theta}\bar{c} \leq \mathrm{h}^0\left(\mathbb{P}^{n-1}, \Omega^{\mathrm{p}}_{\mathbb{P}^{n-1}}(d+p)\right)$. Then $\mathrm{H}\left(\Omega^{\mathrm{p}+1}_{\mathbb{P}^n}(d+p), \Omega^{\mathrm{p}+1}_{\mathbb{P}^{n-1}}(d+p);\ \bar{a}, \bar{b}, \bar{c}\right)$ implies $\mathrm{H}\left(\mathcal{O}_{\mathbb{P}^n}(\mathrm{d}-1)^{\oplus\binom{\mathrm{n}}{\mathrm{p}+1}}, \Omega^{\mathrm{p}}_{\mathbb{P}^{n-1}}(d+p);\ e, f, g\right)$.

Proof. The proof of this lemma follows from application of theorem 2.1 on the exact sequence;

$$0 \longrightarrow \mathrm{H}^0\left(\mathbb{P}^n, \Omega^{\mathrm{p}+1}_{\mathbb{P}^n}(d+p)\right) \longrightarrow \mathrm{H}^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\mathrm{d}-1)^{\oplus\binom{\mathrm{n}}{\mathrm{p}+1}}\right) \longrightarrow \mathrm{H}^0\left(\mathbb{P}^{n-1}, \Omega^{\mathrm{p}}_{\mathbb{P}^{n-1}}(d+p)\right) \longrightarrow 0$$

Lemma 3.8

Consider $\mathrm{H}\left(\mathcal{O}_{\mathbb{P}^n}(\mathrm{d}-1)^{\oplus\binom{\mathrm{n}}{\mathrm{p}+1}}, \Omega^{\mathrm{p}}_{\mathbb{P}^{n-1}}(d+p);\ s_1, s_2, 0\right)$where $d \geq 1$, s_1 and s_2 are non negative integers satisfying $\binom{n}{p+1} s_1 + \binom{n-1}{p} s_2 = h^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\mathrm{d}-1)^{\oplus\binom{\mathrm{n}}{\mathrm{p}+1}}\right)$ and $\binom{n-1}{p} s_2 \leq h^0\left(\mathbb{P}^{n-1}, \Omega^{\mathrm{p}}_{\mathbb{P}^{n-1}}(d+p)\right)$. Suppose that the map $\mathrm{H}^0\left(\mathbb{P}^n, \Omega^{\mathrm{p}+1}_{\mathbb{P}^n}(d+p)|_{S_1}\right) \longrightarrow \mathrm{H}^0\left(\mathbb{P}^n, \Omega^{\mathrm{p}+1}_{\mathbb{P}^n}(d+1)|_{S_1}\right)$ is injective and that the map $\mathrm{H}^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\mathrm{d}-1)^{\oplus\binom{\mathrm{n}}{\mathrm{p}+1}}\right) \longrightarrow \mathrm{H}^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\mathrm{d}-1)^{\oplus\binom{\mathrm{n}}{\mathrm{p}+1}}|_{S_1}\right)$ is surjective where $S_1 \subset \mathbb{P}^n$ is a general set of points, then the hypothesis $\mathrm{H}\left(\mathcal{O}_{\mathbb{P}^n}(\mathrm{d}-1)^{\oplus\binom{\mathrm{n}}{\mathrm{p}+1}}, \Omega^{\mathrm{p}}_{\mathbb{P}^{n-1}}(d+p);\ s_1, s_2, 0\right)$ is true.

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Lemma 3.9

Consider $\mathrm{H}\left(\mathcal{O}_{\mathbb{P}^n}(\mathrm{d}-1)^{\oplus\binom{n}{p+1}}, \Omega^{p}_{\mathbb{P}^{n-1}}(d+p);\ s_1, s_2, 1\right)$ where $d \geq 1$, s_1 and s_2 are non negative integers satisfying $\binom{n}{p+1}s_1 + \binom{n-1}{p}s_2 + \varepsilon = h^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\mathrm{d}-1)^{\oplus\binom{n}{p+1}}\right)$ and $\binom{n-1}{p}s_2 + \varepsilon \leq h^0\left(\mathbb{P}^{n-1}, \Omega^{p}_{\mathbb{P}^{n-1}}(d+p)\right)$. Suppose that the map $\mathrm{H}^0(\mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^n}(d+p)|_{S_1}) \longrightarrow \mathrm{H}^0(\mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^n}(d+1)|_{S_1})$ is injective and that the map $\mathrm{H}^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\mathrm{d}-1)^{\oplus\binom{n}{p+1}}\right) \longrightarrow \mathrm{H}^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\mathrm{d}-1)^{\oplus\binom{n}{p+1}}|_{S_1}\right)$ is surjective where $S_1 \subset \mathbb{P}^n$ is a general set of points, then the hypothesis $\mathrm{H}\left(\mathcal{O}_{\mathbb{P}^n}(\mathrm{d}-1)^{\oplus\binom{n}{p+1}}, \Omega^{p}_{\mathbb{P}^{n-1}}(d+p);\ s_1, s_2, 1\right)$ is true.

Proof. To prove lemma 3.8, consider the short exact sequence below.

$$0 \longrightarrow \mathrm{H}^0\left(\mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^n}(d+p)\right) \longrightarrow \mathrm{H}^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\mathrm{d}-1)^{\oplus\binom{n}{p+1}}\right) \longrightarrow \mathrm{H}^0\left(\mathbb{P}^{n-1}, \Omega^{p}_{\mathbb{P}^{n-1}}(d+p)\right) \longrightarrow 0$$

From the sequence above, we can construct the diagram below;

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & ker\phi \arrow[d] \arrow[r] & \mathrm{H}^0\left(\mathbb{P}^{n-1}, \Omega^{p}_{\mathbb{P}^{n-1}}(d+p)\right) \arrow[d, equal] & \\
0 \arrow[r] & \mathrm{H}^0\left(\mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^n}(d+p)\right) \arrow[r] \arrow[d] & \mathrm{H}^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\mathrm{d}-1)^{\oplus\binom{n}{p+1}}\right) \arrow[r] \arrow[d, "\phi"] & \mathrm{H}^0\left(\mathbb{P}^{n-1}, \Omega^{p}_{\mathbb{P}^{n-1}}(d+p)\right) \arrow[r] & 0 \\
 & \mathrm{H}^0(\mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^n}(d+1)|_{S_1}) \arrow[r] & \mathrm{H}^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\mathrm{d}-1)^{\oplus\binom{n}{p+1}|_{S_1}}\right) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

The map $ker\phi \longrightarrow \mathrm{H}^0\left(\mathbb{P}^{n-1}, \Omega^{p}_{\mathbb{P}^{n-1}}(d+p)\right)$ is injective. Let $W \subseteq \mathcal{O}_{\mathbb{P}^n}(\mathrm{d}-1)^{\oplus\binom{n}{p+1}}$ be the image of this map. By hypothesis $\dim W = s_2$. It then follows from the diagram below

that map $\alpha: W \longrightarrow \mathrm{H}^0\left(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(\mathrm{d}-1)^{\oplus\binom{n}{p+1}}\right)$ is bijective.

$$\begin{tikzcd}
\ker\phi \arrow[r] \arrow[d] & W \arrow[d] \\
\mathrm{H}^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\mathrm{d}-1)^{\oplus\binom{n}{p+1}}|_{S_2}\right) \arrow[r, equal] & \mathrm{H}^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\mathrm{d}-1)^{\oplus\binom{n}{p+1}}|_{S_2}\right)
\end{tikzcd}$$

Consequently, the map $\mathrm{H}^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\mathrm{d}-1)^{\oplus\binom{n}{p+1}}\right) \longrightarrow \oplus_{i=1}^{s_1}\mathcal{O}_{\mathbb{P}^n}(\mathrm{d}-1)^{\oplus\binom{n}{p+1}}|_{S_1} \oplus_{j=1}^{s_2}\mathcal{O}_{\mathbb{P}^{n-1}}(\mathrm{d}-1)^{\oplus\binom{n}{p+1}}|_{S_2}$ is bijective and $\mathrm{H}\left(\mathcal{O}_{\mathbb{P}^n}(\mathrm{d}-1)^{\oplus\binom{n}{p+1}}, \Omega^{p}_{\mathbb{P}^{n-1}}(d+p);\ s_1, s_2, 0\right)$ is true.

Remark 3.10

a. Lemma 3.9 can be proved in a similar manner, except that this time one takes care of the quotients.

b. Lemma 3.7 is useful for the cases when lemma 3.6 fails, that is, when there are few points relative to d. It carries out a shift to reduce the size of d.

c. Lemmas 3.8 and 3.9 are useful to prove cases where lemma 3.7 does not apply.

4. CONCLUSION

The minimal resolution conjecture predicts the form of the minimal free resolution for points in general position in the projective space $\mathbb{P}^n$. It asserts that the free resolution of points in such a configuration has no ghost terms. More prescisely, each of the modules in this free resolution is of the form $F_p = R(-d-p)^{a_{p-1}} \oplus R(-d-p-1)^{b_p}$ with at most one of the betti numbers a_p and b_p being non zero.

One of the methods that can be used to tackle this conjecture is the method of Horace. This is an inductive method and the proof is basically using diagram chasing. To use this method, one needs an elementary transformation. In our case the elementary transformation is given in diagram 3.1. Using the leftmost sequence and the sequence in the middle row, one comes up with the inductive hypotheses. The ultimate goal is to prove that the map below is injective, bijective or surjective.

$$H^0\left(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^{p+1}(d+p+1)\right) \longrightarrow \oplus_{i=1}^{m} \Omega_{\mathbb{P}^n}^{p+1}(d+p+1)|_{P_i} \tag{4.1}$$

Since the betti numbers are given by

$$a_p = max\left\{0, h^0\left(\mathbb{P}^n,\ \Omega_{\mathbb{P}^n}^{p+1}(d+p+1)\right) - rk\left(\Omega_{\mathbb{P}^n}^{p+1}(d+p+1)\right)m\right\}$$

And $b_p = max\left\{0, rk\left(\Omega_{\mathbb{P}^n}^{p+1}(d+p+1)\right)m - h^0\left(\mathbb{P}^n,\ \Omega_{\mathbb{P}^n}^{p+1}(d+p+1)\right)\right\}$

then bijectivity of the map above implies that both a_p and b_p are zero, while injectivity or surjec-tivity implies exactly one of the betti numbers a_p and b_p is zero.

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